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On S_w*-Normal Spaces

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Abstract

The aim of this paper is to introduce and study a new class of spaces, called S_w*- normal spaces. Also, the relationship between this type of space and several other known types of spaces and functions has been dealt with.

Introduction

Throughout this paper by a space (X, τ) (or simply X) means a topological space on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X, the ($Cl_X(A)$ or Cl(A)) and ($Int_X(A)$ or Int(A)) denote the closure and the interior of A respectively, while Sw-Cl(A) and Sw-Int(A) denote the Sw-closure and Sw-interior of A, respectively.

A subset A of a space (X, τ) is called, a semi-open [5] (resp., preopen[6] and α-open [12]) set, if $A\subseteq Cl(Int(A))$ (resp., $A\subseteq Int(Cl((A))$ and $A=Int(Cl(Int(A)))$. The complement of a semi-open (resp., preopen and α-open) set is called semi-closed (resp., preclosed and α-closed). The notion of Sw-open set was introduced in 2009 by Abddullah L. S. [1], and then she defined Sw-normal space by using Sw-open sets which it is a generalization of S_w^*-normal space which we defined it in this paper.

The aim of this paper is to define S_w^*-normal space and we give several characterizations and some properties of it, also we investigate its relationships with other related spaces and functions.

1. Preliminaries

We recall some basic definitions and results which will be used in the next section. We have the following results from [1].

Definition 1.1 A subset A of a topological space (X, τ) together with the empty set is called an **Sw-open set** if $Int(A)\neq\varphi$ The family of all Sw-open sets of X is denoted by SwO(X, τ) or SwO(X).

The complement of an Sw-open set is called an Sw-closed set. A set A is called Sw-regular if it both Sw-open and Sw-closed set.

Proposition 1.2 Let A be any Sw-open set in a space X and let Y be a subspace of X. If $Int_X(A)\cap Y\neq\varphi$, then $A\cap Y$ is an Sw-open set in Y.

Lemma 1.3 If A⊂B and A is an Sw-open set, then so is B.
Definition 1.4 A topological space (X, τ) is called **Sw-regular** if for each closed set F and each x∉ F in X, there are disjoint Sw-open sets U and V such that x∈ U and F⊂V.
Definition 1.5 A topological space (X, τ) is called an **Sw-T_2 space**, if for each pair of distinct points x, y in X, there exist two disjoint Sw-open sets U and V of X containing x and y respectively.
Definition 1.6 A topological space (X, τ) is called **S^*-normal** (= **semi-normal**[7]) (resp., **Sw-normal**[1]) if for each pair of disjoint semi-closed (resp., closed) sets A and B in X, there are disjoint semi-open (resp., Sw-open) sets M and N such that A⊂M and B⊂N.

The following topological concepts and results are found in Dontchev **[6]**.
Definition 1.7 A space X is called **hyperconnected,** if every nonempty open subset of X is dense. A hyperconnected space sometimes is called irreducible.
Lemma 1.8 Let (X, τ) be any topological space. Then the following statements are equivalent:
1) X is hyperconnected,
2) The intersection of any two non-empty open sets is non-empty.

Definition 1.9[8] A topological space (X, τ) is said to be **locally indiscrete** if every open subset of X is closed.
Theorem 1.10[8] A space X is locally indiscrete if and only if every subset of X is preopen.
Lemma 1.11[8] If X is locally indiscrete, then each semi-closed subset of X is closed and hence clopen.
Definition 1.12[10] A space X is called **semi-compact** if every semi-open cover of X admits a finite subcover.
Definition 1.13[9] A space (X, τ) is called **s-regular**, if for each closed set A and each x∉ A, there exist disjoint semi-open sets U and V such that A⊂U and x∈ V.
Definition1.14[11] space X is said to be **extremally disconnected,** if for every open set in X its closure is also open, or equivalently if the interior of every closed set is closed.
Theorem 1.15[12] let (X, τ) be a topological space, then the following statements are equivalent:
1) (X, τ) is extremally disconnected,
2) The collection of all semi-open sets of X forms a topology on X.

Definition 1.16[13] Let A be a subset of a set X. The characteristic function χ_A of a set A from a set X into {0, 1} is defined by: $\chi_{A(x)} = \begin{cases} 1 & if\ x\in A \\ 0 & if\ x\notin A \end{cases}$.

Definition 1.17[1] A function *f* from a space X into a space Y is called Sw-irresolute, if for each x∈ X and for each Sw-open set A in Y containing *f*(x), there exists an Sw-open set U in X containing x such that *f*(U) ⊂A.
Theorem 1.18[1] A function *f* from a space X into a space Y is Sw-irresolute if and only if f^{-1}(A) is Sw-open in X for every Sw-open set A in Y.
Theorem 1.19[1] If *f* is a continuous function from a space X onto a space Y, then *f* is Sw-irresolute.
Definition 1.20[3] Let *f* be a function from a space X into a space Y. *f* is called Somewhat open (briefly Sw-open) if U is a non-empty open subset of X, then there is a non-empty open subset V of Y such that V⊂*f* (U).

Theorem 1.21 A function *f* from a space X into a space Y is:
1) Irresolute if and only if f^{-1}(U) is semi-closed in X for every semi-closed subset U in Y [2].
2) Somewhat open (briefly Sw-open) if and only if Int(*f* (A))≠φ for all A⊂X in which Int(A) ≠φ [3].

2. S_w*-Normal Spaces

Definition 2.1 A topological space (X, τ) is called S_w^*-normal, if for each pair of disjoint sets A and B in X, where A is semi-closed and B is closed, there are disjoint Sw-open sets U and V such that A⊂U and B⊂V.

It is clear that from the above definition that every S^*-normal space is S_w^*-normal, but the converse is not true in general as shown in the following example:

Example 2.2 Let X = {a, b, c, d} with τ = {φ, X, {a}, {b, c}, {a, b, c}, {b, d}, {b}, {a, b}, {a, b, d}, {b, c, d}}. Then (X, τ) is an S_w^*-normal space, but it is not S^*-normal.

The following theorem is a characterization of an S_w^*-normal spaces:

Theorem 2.3 A topological space (X, τ) is S_w^*-normal if and only if for each semi-closed set A and each open set U, which contains A, there exists an Sw-open set M in X such that $A \subset M \subset$ Sw-Cl(M) $\subset U$.

Proof: Let A be any semi-closed subset in an S_w^*-normal space X and U be any open supper set of A in X. Then X\U is closed and A⋂(X\U)= φ. So by the hypothesis there exist two disjoint Sw-open sets L and M such that $A \subset M$, X\U ⊂ L and M⋂L=φ. Since M⋂L=φ, then M ⊂ (X\L) but X\U ⊂ L, then (X\L) ⊂ U and so M ⊂ U. Since M and L are Sw-open sets, then X\M and X\L are Sw-closed sets and so Sw-Cl(X\M)=X\M and Sw-Cl(X\L) = X\L, so A ⊂ M ⊂ Sw-Cl(M) ⊂ Sw-Cl(X\L)= X\L ⊂ U. Thus A ⊂ M ⊂ Sw-Cl(M) ⊂ U.

Conversely, let the condition be satisfied and let A_1, A_2 be two disjoint subsets of X such that A_1 is semi-closed and A_2 is closed. Since $A_1 \cap A_2 = \varphi$. Then $A_1 \subset X \setminus A_2$, where $X \setminus A_2$ is open, so by the hypothesis there exists an Sw-open set M such that $A_1 \subset M \subset$ Sw-Cl(M) $\subset X \setminus A_2$. Put L=X\Sw-Cl(M). Consequently there exist two disjoint Sw-open sets L and M such that $A_1 \subset M$ and $A_2 \subset L$. Therefore X is an S_w^*-normal space.

As a corollary of Theorem 2.3 we have:

Corollary 2.4 A space X is S_w^*-normal if and only if for each semi-closed set F and each open set U containing F, there exists a subset A of X, such that F ⊂ Sw-Int(A) ⊂ Sw-Cl(A) ⊂ U.

Proof: It follows from (Theorem 2.3 and Lemma 1.3).

Proposition 2.5 Every semi-T_1 S_w^*-normal space is an Sw-regular space.

Proof: Let A be any closed subset in an S_w^*-normal space X and x∈ X such that x∉ A. Since X is a semi-T_1 space, then {x} is a semi-closed subset of X with {x}⋂A=φ. By the S_w^*-normality of X, there exist two disjoint Sw-open sets M and L of X such that {x} ⊂ M, A ⊂ L. So x∈ M, A ⊂ L. Thus X is an Sw-regular space.

From Proposition 2.5 we have the following corollary:

Corollary 2.6 Every τ_1 S_w^*-normal space is an Sw-regular space.

Proof: It follows from (Proposition 2.5).

Proposition 2.7 Every T_1 S_w^*-normal space is Sw-T_2.

Proof: Let x and y be any two distinct points in X. Since X is a T_1-space, then {x} and {y} are closed subsets of X with {x}⋂{y}=φ. By S_w^*-normality of X there exist two disjoint Sw-open sets U and V of X such that {x}⊂Uand {y}⊂V. So x ∈ U, y ∈ V. Thus X is an Sw-T_2 space.

The following theorem shows the relations between S_w^*-normal and s-regular space when a space is semi-compact and extremelly disconnected.

Theorem 2.8 If X is an s-regular, semi-compact and extremally disconnected space, then it is an S_w^*-normal space.

Proof: Let F and H be any two disjoint subsets of a space X, where F is semi-closed and H is closed. Since X is s-regular, then for each x∈F (means x ∉H), then there exist two semi-open sets U_x and V_x of X such that x∈U_x, H⊂V_x and $U_x \cap V_x = \varphi$. The collection {U_x: x∈F} is a semi-open cover of F and since F is semi-closed then X\F is semi-open. This implies that M={U_x: x∈F}⋃(X\F) is a semi-open cover of X. Since X is semi-compact, then there exist a finite sub family of M such that X=$(\bigcup_{i=1}^{n} U_{xi})$⋃(X\F) and then, F⊂$\bigcup_{i=1}^{n} U_{xi}$. Let U=$\bigcup_{i=1}^{n} U_{xi}$ and V=$\bigcap_{i=1}^{n} V_{xi}$. Since X is extremally disconnected, then U and V are disjoint semi-open sets and so they are Sw-open sets in X, such that F⊂U and H⊂V. That X is an S_w^*-normal space.

Remark 2.9 The property of an S_w^*-normal space is not a hereditary property. As seen in Example 2.2, (X, τ) is an S_w^*-normal space, but Y= {b, c, d} is a subset of X and so τ_y={φ,Y,{b,c},{b,d},{b}}. Then {d} and {c} are disjoint closed set and so one of them say {d} is a semi-closed subset of Y, but there does not exist two disjoint Sw-open sets in Y containing them respectively. Therefor (Y, τ_y) is not S_w^*-normal.

The following result is a characterization of an S_w^*-normal spaces:

Theorem 2.10 For a topological space (X, τ) the following statements are equivalent:

1. X is an S_w^*-normal space.
2. For every two subsets U and V such that U is open and V is semi-open whose union is X, there exists Sw-closed sets A and B of X, such that $A\subset U$, $B\subset V$ and $A\bigcup B=X$.

Proof:(1)$\Rightarrow$(2) Let U and V be any two subsets in an S_w^*-normal space X, where U is open and V is semi-open such that $X=U\bigcup V$. Then X\U is closed and X\V is semi-closed which are disjoint. So by S_w^*-normality of X, there exist disjoint Sw-open sets U_1 and V_1 such that $X\setminus U\subset U_1$ and $X\setminus V\subset V_1$, so $X\setminus U_1\subset U$ and $X\setminus V_1\subset V$. Let A= $X\setminus U_1$ and B = $X\setminus V_1$. Then A and B are Sw-closed sets such that $A\subset U$, $B\subset V$ and $A\bigcup B = X$.

(2)$\Rightarrow$(1) Let U and V be two disjoint subsets of X such that U is closed and V is semi-closed. Then X\U is open and X\V is semi-open. Since U ∩ V =φ, then (X\U) ∪ (X\V) = X. So there exist two Sw-closed sets A and B of X such that $A\subset X\setminus U$, $B\subset X\setminus V$ and $A\bigcup B = X$. Then $U\subset X\setminus A$ and $V \subset X\setminus B$, where X\A, X\B are Sw-open sets, further more X\A∩X\B =X\(A∪B)=X\X=φ. Therefore (X, τ) is an S_w^*-normal space.

Recall that a subset A of space X is called a **generalized semi-closed** (briefly gs-closed) if sCl(A)⊂ U,whenever A ⊂ U and U is an open set in X [2].

By using a gs-closed set, we conclude the following result:

Theorem 2.11 Let (X, τ) be a space and $F\bigcap A=\varphi$, where F is closed and A is a gs-closed set in X, then (X, τ) is S_w^*-normal if and only if $F\bigcap sCl(A) =\varphi$, further, there exist disjoint Sw-open sets U and V of X such that $A\subset U$ and $F\subset V$.

Proof: Let X be an S_w^*-normal space. Since $A\bigcap F =\varphi$, then $A\subset X\setminus F$ where X\F is open. Since A is a gs-closed set, then $sCl(A)\subset X\setminus F$. That is $sCl(A)\bigcap F =\varphi$, so by S_w^*-normality of X, there exist two disjoint Sw-open sets U and V of X such that $F\subset U$ and $sCl(A)\subset V$, but $A\subset sCl(A)\subset V$. Thus, $A\subset V$, $F\subset U$ and $U\bigcap V =\varphi$.

Conversely, let A and B be two disjoint sets in X, where A is semi-closed and B is closed. Since every semi-closed set is gs-closed, then by the hypothesis, there exist disjoint Sw-open sets U and V such that $A \subset U$ and $B\subset V$. Thus X is an S_w^*-normal space.

Like the well-known Urysohn's Lemma we have the following result, although we prove it in a different way:

Theorem 2.12 A space X is an S_w^*-normal if and only if for any two disjoint subsets A and B of X, such that A is closed and B is semi-closed, there exists an Sw-irresolute function $f: X\rightarrow[0, 1]$ such that $f(x) = 0$ for all $x\in A$ and $f(x) = 1$ for all $x\in B$.

Proof: Let X be an S_w^*-normal space and A, B be any disjoint subsets of X such that A is closed and B is semi-closed. Then there exist two disjoint Sw-open sets U and V such that $A\subset U$ and $B\subset V$. To show that there exists an Sw-irresolute function $f: X\rightarrow[0, 1]$ such that $f(x) = 0$ for all $x\in A$ and $f(x) =1$ for all $x\in B$. Consider the characteristic function $\chi_{Sw-Cl(V)}$. Then $\chi_{Sw-Cl(V)}^{-1}(a, 1]$ = Sw-Cl(V), where Sw-Cl(V) is an Sw-regular set and $\chi_{Sw-Cl(V)}^{-1}[0, a)$ = X\Sw-Cl(V) which is also an Sw-regular set. Also $\chi_{Sw-Cl(V)}^{-1}(a, b)= \varphi$ where $0<a<b<1$ and $\chi_{Sw-Cl(V)}^{-1}[0, 1] = X$. Thus $\chi_{Sw-Cl(V)}$ is an Sw-irresolute function, such that $\chi_{Sw-Cl(V)}(A) = \{0\}$ and $\chi_{Sw-Cl(V)}(B) = \{1\}$.

Conversely, suppose that, for any two disjoint subsets A and B of X such that A is closed and B is semi-closed of X an irresolute function $f: X \to [0, 1]$ such that $f(x)=0$ for all $x\in A$ and $f(x)=1$ for all $x\in B$. To show X is an S_w^*-normal space, let G be an Sw-open set of [0, 1] containing 0 but not containing 1. Then $f^{-1}(G)$ is an Sw-open set in X such that $A\subset f^{-1}(G)$. Again let H be an Sw-open set of [0, 1] containing 1 but not 0 such that $G\cap H=\varphi$, then $f^{-1}(H)$ is an Sw-open set in X such that $B\subset f^{-1}(H)$. Let $f^{-1}(G)=M$ and $f^{-1}(H)=N$. Thus M and N are Sw-open sets in X such $A\subset M$, $B\subset N$ and $M\cap N=\varphi$. Therefore X is an S_w^*-normal space.

Proposition 2.13 The property of a space being an S_w*-normal space is preserved under an onto, irresolute continuous and Sw-open function.

Proof: Let A and B be two disjoint subsets of Y such that A is closed and B is semi-closed. Then $f^{-1}(A)$ is closed and $f^{-1}(B)$ is semi-closed sets in X, so by S_w^*-normality of X, there exist two disjoint Sw-open sets U and V such that $f^{-1}(A)\subset U$ and $f^{-1}(B)\subset V$. Since f is an Sw-open function and Int(U) $\neq\varphi$, Int(V) $\neq\varphi$, then Int(f(U)) $\neq\varphi$ and Int(f(V)) $\neq\varphi$. That is , f(U) and $f(V)$ are Sw-open sets in Y. Since $f^{-1}(A)\subset U$ and $f^{-1}(B)\subset V$, then $A= f(f^{-1}(|A))\subset f(U)$ and $B= f(f^{-1}(B))\subset f(V)$. Thus $A\subset f(U)$ and $B\subset f(V)$ and $f(U)\cap f(V)= \varphi$. Hence Y is an S_w^*-normal space.

Theorem 2.14[4] Let X and Y be any two topological spaces, if $f: X\to Y$ is continuous and open, then it is irresolute.

Corollary 2.15 The property of a space being an S_w*-normal space is preserved under an onto, continuous and open function. That is, S_w^*-normal space is a topological property.

Proof: Its proof follows directly from the Proposition2.13, Theorem2.14 and from the fact that each open function is Sw-open.

Theorem 2.16 Let f be a function from a space X onto a space Y, such that f is continuous and Sw-open function with gs-closed point inverses. If X is an S_w^*-normal space, then Y is Sw-regular.

Proof: Let A be any closed subset of Y and $y\in Y$, such that $y\notin A$. Since f is continuous. Then $f^{-1}(A)$ is closed in X and by the hypothesis $f^{-1}(\{y\})$ is a gs-closed set in X. Clearly $f^{-1}(A)$ and $f^{-1}(\{y\})$ are disjoint. Then by Proposition 2.12, there exist two disjoint Sw-open sets U and V such that $f^{-1}(\{y\})\subset U$ and $f^{-1}(A)\subset V$, and then $f(f^{-1}(\{y\}))\subset f(U)$ and $f(f^{-1}(A))\subset f(V)$. Therefore, $y\in f(U)$, $A\subset f(V)$ and $f(U)\cap f(V)= \varphi$. Also f(U) and f(V) are Sw-open sets in Y. That is, Y is an Sw-regular space.

Corollary 2.17 If f is a homeomorphism from an S_w^*-normal space X onto a space Y with gs-closed point inverses, then Y is an Sw-regular space.

Proof: Follows from Theorem 2.16 and from the fact that each open function is Sw-open.

Recall that a function $f: X\to Y$ is **presemi-closed [5],** if the image of each semi-closed set in X is a semi-closed set in Y.

Theorem 2.18 If the codomain of a closed, presemi-closed and Sw-irresolute injective function is an S_w^*-normal space, then so is its domain.

Proof: Let f be a function, from a space X into an S_w^*-normal space Y, and let V_1 and V_2 be two disjoint subsets in X such that V_1 is closed and V_2 is semi-closed. Then $f(V_1)$ and $f(V_2)$ are disjoint closed and semi-closed subsets in Y, respectively. Since Y is an S_w^*-normal space, then there exist two disjoint Sw-open sets A and B in Y, such that $f(V_1)\subset A$, $f(V_2)\subset B$. Then $V_1 = f^{-1}(f(V_1))\subset f^{-1}(A)$ and $V_2 = f^{-1}(f(V_2))\subset f^{-1}(B)$. Since f is Sw-irresolute, then $f^{-1}(A)$ and $f^{-1}(B)$ are Sw-open sets in X such that $V_1\subset f^{-1}(A)$, $V_2\subset f^{-1}(B)$ and $f^{-1}(A)\cap f^{-1}(B)= \varphi$. Thus X is an S_w^*-normal space.

Corollary 2.19 If f is a closed, presemi-closed and continues function, from a space X onto an S_w^*-normal space Y, then X is also S_w^*-normal.

Proof: Follows from Theorem 1.19 and Theorem 2.18.

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